APPENDIX I

MATRIX ALGEBRA

For those with inadequate knowledge of the fundamentals of matrix algebra, a brief introduction is given here.

Definition of a Matrix

In the most general sense a matrix is a rectangular array of numbers, or symbols for numbers, which may be combined with other such arrays according to certain rules. When a matrix is written out in full, it has an appearance of which the following is typical:

$$\begin{bmatrix} 4 & -7 & 6 & 0 \\ 2 & 9 & -1 & -8 \\ 2 & 0 & 5 & 4 \\ -8 & 7 & 0 & -3 \\ 6 & 3 & -4 & 7 \end{bmatrix}$$

Note the use of square brackets to enclose the array; this is a conventional way of indicating that the array is to be regarded as a matrix (instead of, perhaps, as a determinant).

In order to discuss matrices in a general way, certain general symbols are commonly used. Thus we may write a symbol for an entire matrix as a script letter, for example, $\mathscr{A}$, which stands for

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ & & & \vdots & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

the three matrices must be of the same dimensions. The elements of $\mathscr{C}$ are given by

$$c_{pq} = a_{pq} \pm b_{pq}$$

A matrix may be multiplied by a scalar number or by another matrix. For multiplication of a matrix $[c_{ij}]$ by a scalar, α, we have

$$\alpha[c_{ij}] = [\alpha c_{ij}] = [c_{ij}\alpha] = [c_{ij}]\alpha$$

Multiplication of a matrix by a matrix is somewhat more complicated. In the first place, it can be done only if the two matrices are *conformable.* This means that, if we wish to take the product $\mathscr{A}\mathscr{B} = \mathscr{C}$, the number of columns in $\mathscr{A}$ must be equal to the number of rows in $\mathscr{B}$. If this requirement is satisfied, so that $\mathscr{A}$ is of order $(n \times h)$ while $\mathscr{B}$ is of order $(h \times m)$, then $\mathscr{C}$ will be of order $(n \times m)$. Each element of the product matrix is given by the following expression:

$$c_{il} = \sum_k a_{ik} b_{kl} \tag{AI-1}$$

This sum may be written out explicitly as follows:

$$c_{il} = a_{i1}b_{1l} + a_{i2}b_{2l} + a_{i3}b_{3l} + a_{i4}b_{4l} + \cdots + a_{ih}b_{hl}$$

where a_{ih} is the last element in the ith row of $\mathscr{A}$, and b_{hl} is the last element in the lth column of $\mathscr{B}$. Perhaps this will be still clearer if we explicitly write out the process of multiplying a 3×2 matrix into a 2×4 matrix.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \end{bmatrix}$$

$$c_{11} = a_{11}b_{11} + a_{12}b_{21} \qquad c_{21} = a_{21}b_{11} + a_{22}b_{21}$$

$$c_{12} = a_{11}b_{12} + a_{12}b_{22} \qquad c_{22} = a_{21}b_{12} + a_{22}b_{22}$$

$$c_{13} = a_{11}b_{13} + a_{12}b_{23} \qquad c_{23} = a_{21}b_{13} + a_{22}b_{23}$$

$$c_{14} = a_{11}b_{14} + a_{12}b_{24} \qquad c_{24} = a_{21}b_{14} + a_{22}b_{24}$$

$$c_{31} = a_{31}b_{11} + a_{32}b_{21}$$

$$c_{32} = a_{31}b_{12} + a_{32}b_{22}$$

$$c_{33} = a_{31}b_{13} + a_{32}b_{23}$$

$$c_{34} = a_{31}b_{14} + a_{32}b_{24}$$

Suppose we wish to reflect this point x, y, z through the origin. Its coordinates will now be $-x$, $-y$, $-z$. We may express this by the following matrix equation

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -x \\ -y \\ -z \end{bmatrix}$$

If we want to rotate the point around the z axis by $\pi/2$, or reflect it through the xy plane we can write

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -x \\ -y \\ -z \end{bmatrix} = \begin{bmatrix} y \\ -x \\ -z \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} y \\ -x \\ -z \end{bmatrix} = \begin{bmatrix} y \\ -x \\ z \end{bmatrix}$$

If we want to do all of these things, one after the other, we may obtain the matrix to do it by multiplying the individual matrices to get one matrix that does it all. Since matrix multiplication is associative it does not matter how we choose the pairs. Thus, we may write

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We can now apply this to the original point and get, as before,

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ -x \\ z \end{bmatrix}$$

Evaluation and Expansion of Determinants

To find the inverse of a matrix we shall have to employ the corresponding determinant. Determinants are, by definition, square. They consist of a square terms are products of elements on lines running from upper right to lower left. For determinants of the fourth and higher orders the number of products (24 for $n = 4$) exceeds the number which may be enumerated in this way (8 for $n = 4$), and more labor is required to write down the complete expansion of the determinant.

Determinants of order $\geqslant 4$ may conveniently be evaluated by the *method of cofactors*. Inspection of the list of six products whose algebraic sum is the value of a determinant of order 3 shows that we may rewrite it in the follov.-ing way:

$$a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{23}a_{31} - a_{21}a_{33}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Each of the terms in parentheses is the expanded form of the determinant made up of the elements of the original determinant which remain after we strike from it the elements belonging to the row and column of the element in front of the parenthesis. It is given a + sign if the sum of the indices of the element before it is even and a negative sign if the sum of these indices is odd. The terms in the parentheses are called the *cofactors* of the elements in front of the parentheses. Thus we see that the third-order determinant can be evaluated by finding the sum of the products of each element in the first row with its cofactor.

A little reflection will show that we could just as well have arranged the six terms in the expansion so as to have a sum of the products of *any* row or *any* column with their cofactors. For instance, choosing the second column, we can write

$$a_{12}(a_{23}a_{31} - a_{21}a_{33}) + a_{22}(a_{11}a_{33} - a_{13}a_{31}) + a_{32}(a_{13}a_{21} - a_{11}a_{23})$$

It should also not be difficult to see that a similar process can be carried out on a determinant of any order. Thus, using A^{ij} to represent the cofactor of a_{ij}, we can write

$$|A| = \sum_{\substack{i \\ (\text{for any } j)}} a_{ij}A^{ij} = \sum_{\substack{j \\ (\text{for any } i)}} a_{ij}A^{ij}$$

One further important property of any determinant is that *if any two rows or columns are identical the value of the determinant is zero.* This is easily proved. Suppose that the pth and rth rows are identical. Then for any term in the expansion, say

$$a_{l1}a_{m2}a_{n3} \cdots a_{pi}a_{rj} \cdots$$

there must be another which is identical except that it will contain a_{ri} and a_{pj}. Now suppose that the column indices are arranged serially in the term shown element is the sum of products of the elements of a certain row, say the ith, with the cofactors of the elements of some other row, say the jth. Such a sum is, in fact, the expansion in the elements of the ith row with their cofactors of a determinant in which the ith and jth rows are identical. Since we have already seen that the value of such a determinant must be zero, all off-diagonal elements of the product $\mathscr{A}\hat{\mathscr{A}}$ are zero. It is also easy to see that $\mathscr{A}\hat{\mathscr{A}} = \hat{\mathscr{A}}\mathscr{A}$.

Thus we have the result

$$\mathscr{A}\hat{\mathscr{A}} = \hat{\mathscr{A}}\mathscr{A} = \begin{bmatrix} |A| & 0 & 0 & 0 & \cdots & 0 \\ 0 & |A| & 0 & 0 & \cdots & 0 \\ & & & & \vdots & \\ 0 & & & & \cdots & |A| \end{bmatrix}$$

$$= |A| \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ & & & & \vdots & \\ 0 & & & & \cdots & 1 \end{bmatrix}$$

$$= |A|\,\mathscr{E}$$

Now, referring to the definition of $\mathscr{A}^{-1}$, we see that

$$\mathscr{A}^{-1} = \frac{\hat{\mathscr{A}}}{|A|}$$

That is, each element of $\mathscr{A}^{-1}$ is the element of $\hat{\mathscr{A}}$ divided by $|A|$. Since division by zero is not defined, only matrices for which the corresponding determinants are nonzero can have inverses. A matrix $\mathscr{A}$ such that $|A| = 0$ is said to be *singular* (no inverse), whereas matrices of which the corresponding determinants are nonzero are said to be *nonsingular*. Only nonsingular matrices can occur in representations of a group. It is also clear that since only square matrices can have corresponding determinants, only square matrices can have inverses.